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Pareto Optimal Policies for Harvesting with Multiple Objectives

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ABSTRACT

The standard recursive function formulation of dynamic programming is shown to extend readily to a problem with multiple objectives. A stochastic model is formulated which is relevant to capital accumulation models, renewable resource management, and economic growth. For relevant special cases, it is proven that the knowledge of the single objective optimal policy functions is sufficient to completely describe the set of dynamically Pareto optimal (or nondominated) policy functions for the multiple objective problem. It is also shown that there is a single policy which is optimal for three related problems: When each objective is given even weighting; when the total "regret" is minimized; and when the maximum regret is minimized.

I. INTRODUCTION

Most models of stochastic decisionmaking concentrate on optimal policies for single objectives, while most decisionmaking situations involve multiple, often conflicting objectives. While single objective optimization can be informative to the decisionmaker, it does not provide any idea of the tradeoffs involved. In this paper the usual dynamic programming formulation for stochastic models is shown to readily extend to stochastic models with multiple objectives. This fact is then used to characterize the dynamic Pareto optimal set for a class of models relevant to capital accumulation and the harvesting of renewable resources.

A related extension of dynamic programming to partially ordered sets can be found in Brown and Strauch [2]. At the time an earlier version of this paper was written, Henig [6] was extending the contraction mapping formulation of Denardo [3] to multiple objectives.

The impetus for this paper comes from the recently enacted Fishery Conservation and Management Act (FCMA), which requires fishery managers to consider biological, social, and economic factors when determining an optimal yield. Usually these factors have conflicting goals, yet a single "best" policy must be determined. A starting procedure would be to

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find some desirable subset of policies, and then consider the tradeoffs between these policies. Pareto optimal, or "efficient," policies are such a class of policies.

I have shown previously [9–11] that stochastic harvesting models are structured capital accumulation models that have structured optimal policies. Similar structure in the multiple objective problem make it possible to describe the set of efficient policies solely from knowing the single objective optimal policy functions. This "uncoupling" of a k-objective problem into k single objective problems greatly increases computational efficiency, and adds insight into the structure of the set of efficient policies.

It is also possible, for this class of harvesting models, to find several policies of interest within the set of efficient policies with little extra effort. Let the k-vector J^* be the optimum optimorum vector, and let the k-vector J be the expected return from a given policy. That is, J^{*i} is the optimal expected return in a single objective problem with objective i as the sole objective. The "p-regret function" $R_P(J)$ [14] is defined as

$$R_P(J)=\begin{cases}\left(\sum_{i=1}^{k}(J^{*i}-J^i)^p\right)^{1/p}, & p<\infty,\\ \max_i (J^{*i}-J^i), & p=\infty,\end{cases}$$

and it is desired to minimize $R_p(J)$ for a fixed value of p. Attention is given to $p=1$, or (the total group regret problem) and $p=\infty$ (the Chebyshev, or minimum maximal, regret problem). A third policy of interest maximizes the unweighted sum of the two objectives. This will be termed the "equitable policy." The stochastic models of this paper have the interesting property that a policy that is equitable also minimizes $R_1(\cdot)$ and $R_\infty(\cdot)$.

II. NOTATION AND DEVELOPMENT

A Markov process is observed for T periods, $T<\infty$. The periods are subscripted by t, $1<t<T$. Alternatively, $n=T-t+1$, the number of periods remaining till the end of the planning horizon, is used to subscript the period. At the start of each period a state $x\in X$ is observed; a decision y is chosen, which may be constrained by the state to lie in some set $Y(x)$; and the random transition to a state next period is given by

$$x_{t+1}=s[x_t, y_t, D_t],$$

where $D_1, D_2,\ldots, D_T$ are independent, identically distributed random variables distributed as the generic random variable D.

In each period, if state x is observed and decision $y\in Y(x)$ is chosen, k one-period rewards are received, given by $G_t(x, y)=(g_t^1(x, y),$

$g_t^2(x, y), \ldots, g_t^k(x, y))$. The ith reward is discounted by a factor α_i, $0 \leqslant \alpha_i \leqslant 1$.

Let $\delta_t(x)$ be a policy function in period t, that is, δ_t maps each $x \in X$ into $Y(x)$. An n-period policy π_n is a sequence of n policy functions, that is, $\pi_n = (\delta_n, \delta_{n-1}, \ldots, \delta_1)$.

A t-period history H_t is a sequence of feasible states, actions, and random variables

$$H_t = (x_1, y_1, D_1; x_2, y_2, D_2; \ldots; x_{t-1}, y_{t-1}, D_{t-1}; x_t)$$

The expected value of an n-period policy π_n, given a history H_{T-n+1}, is defined as

$$v_{\pi_n}(H_{T-n+1}) = E_{\pi_n}\left\{ \sum_{t=T-n}^{T} (\alpha^{t-1} G_t(x_t, y_t)) \middle| H_{T-n+1} \right\}, \tag{2.1}$$

where the addition of the vector return is coordinate by coordinate, and $\alpha^{t-1} G_t$ denotes coordinate by coordinate multiplication of g_t^i by α_i^{t-1}. Define the set V_T as

$$V_T = \{ v_{\pi_T}(x_1) : \pi_T \text{ is a feasible } T\text{-period policy} \}.$$

Then π_T^* is termed *efficient* or *Pareto optimal* if there does not exist some other policy π_T with $v_{\pi_T}(x_1) \in V_T$ such that [1]

$$v_{\pi_T}(x_1) \geqslant v_{\pi_T^*}(x_1). \tag{2.2}$$

Let $\Lambda_T = \{ v_{\pi_T} \in V_T : \pi_T \text{ is an efficient policy} \}$. The vmax, or vector maximization operator, is defined as

$$\text{vmax} : V_T \to \Lambda_T.$$

At times, a slightly abused use of vmax denotes finding efficient policies rather than efficient vectors. However, the usage should be clear from the context.

With these definitions, the n-period problem is

$$\begin{aligned} \text{vmax}\, v_{\pi_T}(x_1) = \text{vmax}\, E&\left(\sum_{t=1}^{T} \alpha^{t-1} G_t(x_t, y_t) \right) \\ \text{s.t.} \quad & x_{t+1} = s[x_t, y_t, D_t], \\ & x_t \in X, \quad y_t \in Y(x). \end{aligned} \tag{2.3}$$

[1] When comparing two k-vectors, $x = (x^1, x^2, \ldots, x^k)$ and $y = (y^1, y^2, \ldots, y^k)$, $x = y$ implies $x^i = y^i$ for all i; $x \geqslant y$ implies $x^i \geqslant y^i$; $x \geqslant y$ implies $x \geqslant y$ but $x \neq y$; and $x > y$ implies $x^i > y^i$ for all i.

Let $f_n^{\text{eff}}(x)$ denote the set of n-period efficient return vectors when x is the observed state, and let $\oplus$ denote the addition of a given vector to every vector in a set of vectors. Thus $G_{n+1}(x,y)\oplus f_n^{\text{eff}}(x)$ adds the vector $G_{n+1}(x,y)$ to each vector in $f_n^{\text{eff}}(x)$. Theorem 2.1 proves that $f_n^{\text{eff}}(x)$ satisfies the standard dynamic programming recursive equations.

THEOREM 2.1

If $G_t(x,y)$ is bounded for every t, then in (2.3):

(i) *If a policy π_n^* is n-period efficient, then any k-period policy π_k such that $\pi_n^* = \{\delta_n, \delta_{n-1}, \dots, \delta_{n-k}, \pi_k\}$ is k-period efficient given the history H_{T-k+1}.*

(ii) *An n-period efficient policy depends on the past history only through the present state and expected efficient values, that is,*

$$v_\pi(H_t) = v_\pi(x_t).$$

(iii)

(a) *If there exists a nonempty set of dynamic efficient policies, then every efficient policy is a solution to the recursive equation*

$$f_0^{\text{eff}}(\cdot) \equiv 0,$$
$$f_n^{\text{eff}}(x) = \text{vmax}\{G_n(x,y) \oplus \alpha E f_{n-1}^{\text{eff}}(s[x,y,D]: x \in X, y \in Y(x)\}. \tag{2.4}$$

(b) *If there exists a sequence of point to set functions $A_1, A_2, \dots, A_T$ such that $A_i: X \to Y(x)$ and the set of resulting policies solve* (2.4), *then the sequence of functions $\{A_i(x)\}$ describe the set of dynamic Pareto optimal policies.*

The system of recursive equations imply that the set of dynamic efficient policies can be found by

(i) choosing an efficient decision in period n,
(ii) following any efficient policy from there onwards.

This is a subtle point, for in Sec. III it will be shown that for a class of problems, the weightings given to each objective by an efficient policy can vary through time.

III. APPLICATIONS

In this section, the system of recursive equations (2.4) and related systems of equations are used to describe dynamic efficient policies for two harvesting or consumption models. In the first model, a single stock of fish (or consumption good) is valued differently by two different user groups. This occurs in a fishery context when the government is the management that must set a quota on a fishing industry, but recreational fishermen or other interest groups also value the standing stock as an important resource. Such a situation arises in the management of the northern anchovy off the coast of California.

The second model is a single stock that is managed by a single agency, where harvesting or consumption each year must be allocated between different user groups. The agency must decide not only the total amount to be allocated, but also how much of the total allocation should be given to each user group. This model differs from related economic work in that the model is dynamic and stochastic.

Lemma 3.1 is due to Geoffrion [4], and is used throughout this section.

LEMMA 3.1

Let $f_1(x), f_2(x), \ldots, f_k(x)$ be concave functions that map a subset of $\mathbb{R}^n$ into R^1. Then the following two statements are equivalent:

(i) *x^* is an argument where*

$$\operatorname{vmax}(f_1(x), f_2(x), \ldots, f(k)) \quad s.t. \quad x \in X$$

obtains a solution.

(ii) *x^* is an argument where*

$$\max \sum_{i=1}^{k} \lambda^i f_i(x)$$

$$s.t. \quad \sum_{i=1}^{k} \lambda^i = 1, \quad 0 \leqslant \lambda^i \leqslant 1 \text{ for all } i$$

obtains a solution for some $\lambda = (\lambda^1, \lambda^2, \ldots, \lambda^k)$. ■

Let x_t denote the population size at the beginning of period t, and let y_t be the population size remaining after harvesting has ceased. The transition to x_{t+1} depends only on y_t and the random variable D_t, that is,

$$x_{t+1} = s[y_t, D_t].$$

It is assumed that:

(i) $s[\cdot, d]$ is concave and continuous for each realization d of D_t. Each user group's one-period return or utility is assumed to be a function of the amount harvested, $x_t - y_t$. This is a simplifying assumption, as for some fisheries the cost of fishing varies greatly with the amount available at the beginning of the period. However, most of the results do not depend on this assumption so much as on the following two assumptions, which are assumed to be valid for $g^i(\cdot)$ for each user group i.

(ii) $g^i(\cdot)$ is concave, bounded, and continuous on the set $C=\{(x,y): x\in X, 0\leq y\leq x\}$

(iii) $g^i(\cdot)$ is nondecreasing. (For the more general case, an additional assumption that $g^{[2]}(\cdot,y)$ is nondecreasing in x is needed.)

These assumptions are discussed in more detail in [9–11].

With n periods remaining till the end of the planning horizon, and for any $x\in X$, assume that all k single objective policy functions $A_n^1(x), A_n^2(x),\ldots, A_n^k(x)$ are known, and reorder the index such that

$$A_n^1(x)\leq A_n^2(x)\leq\cdots\leq A_n^k(x).$$

Let $A_n(x,\lambda)$ be an optimal policy in period n for $x\in X$ and given weights λ.

THEOREM 3.1

Assumptions (i)–(iii) *imply for the problem given in* (2.4) *that for each n and every* $x\in X$:

(i) *A decision y is part of an efficient policy if and only if*

$$A_n^1(x)\leq y\leq A_n^k(x).$$

(ii) *The set of dynamic efficient policies is to choose any y as in* (i), *and then follow any* $(n-1)$*-period efficient policy thereafter.*

Theorem 3.1 decomposes the multiple objective problem into k single objective problems. Moreover, since every solution implies a relative weighting to the different objectives, part (ii) of the theorem states that there exist dynamic efficient policies whose relative weighting can shift through time. This at first seems unreasonable, but two explanations can be given to this fact. A shift of weighting can be thought of as an adjustment to the realization of the stochastic process, rather than to its expected value. The realization may overly favor one group over another, and a shift in weightings can serve as a correction.

Secondly, a shift in weightings can be viewed as a learning process. A decisionmaker's preference may in fact switch as he or she learns more about the actual consequences of their decisions.

If the largest and the smallest values of the set of efficient policies greatly differ in value, then the set of efficient policies does not lend much insight. An initial starting point for a good decision might be $y=(1/k)[\sum_{i=1}^k A_n^i(x)]$. Corollary 3.1 suggests a second policy with desirable properties.

COROLLARY 3.1

The assumptions of Theorem 3.1 *imply that for each n and every* $x \in X$, *an equitable policy also minimizes the total group regret and minimizes the maximum regret.*

Proof. The equitable policy by definition maximizes the total even weighted sum of the objectives. Therefore it must minimize the total deviation from the optimum optimorum vector.

Zangwill [15] proves that if k functions $g_1(x), g_2(x), \ldots, g_k(x)$ are pseudoconcave, then necessary and sufficient conditions in order to

$$\max_x \min_i \{g_i(x)\}$$

are

$$\sum_{i=1}^{k} \nabla^i g^i(x) \cdot \omega^i = 0, \qquad \omega^i \geqslant 0,$$

$$\sum_i \omega^i > 0.$$

$R_\infty(J)$ is to

$$\min_x \max_i \{J_i^* - J_i\},$$

where $J_i^* - J_i$ is convex. This is equivalent to

$$\max_x \min_i \{J_i - J_i^*\},$$

where $J_i - J_i^*$ is concave.

The above necessary and sufficient conditions, however, are identically the Kuhn-Tucker conditions for the dynamic programming problem [8] with equal weights. ∎

The second model concerns the dynamic allocation of a catch quota between user groups. Let z_t^i be the amount of the catch allotted to group i in period t. Assume each user group's one-period return satisfies assumptions (i)–(iii), as well as

(iv) $g_t^i(z_t) = g_t^i(z_t^i)$, $i = 1, \ldots, k$,

that is, each user group's return depends only on the amount that it harvests, independent of the one-period harvest of the other groups.

Assume $z_n^1(x), \ldots, z_n^k(x)$, are optimal policy functions for each group when they are the only one considered, and for convenience the numbering will always be such that they are in ascending order.

THEOREM 3.2

For each n and every $x \in X$, *assumptions* (i)–(iv) *imply the following*:

(i) $z_n = (z_n^1, \ldots, z_n^k)$ *is part of an efficient policy if and only if* $z_n^1(x) \leqslant \sum_{i=1}^k z_n^i \leqslant z_n^k(x)$.

(ii) *For any fixed total harvest* z, *let* z^* *be the solution to*

$$\text{maximize} \quad \sum_{i=1}^{k} \lambda^i g^i(z^i)$$

$$\text{s.t.} \quad \sum_{i=1}^{k} z^i = z; \quad \sum_{i=1}^{k} \lambda^i = 1, \; 0 \leqslant \lambda^i \leqslant 1. \tag{3.2}$$

Then the dynamic recursion can be decomposed into finding an optimal $z = \sum_{i=1}^k z^i$, *and allocating the total according to* (3.2).

(iii) *A dynamic efficient policy is to choose* z_n *as in* (i), *and following any efficient policy till the end of the planning horizon.*

Theorem 3.2 has intuitive economic appeal. It states that while the total quota may vary according to the state observed and the period, the allocation, given a fixed quota, is the same in each period.

Theorem 3.2 also states that it is not desirable *in the long run* to have a quota smaller than that which is the smallest desired for any single user, nor larger than that desired by any user. This is true, even though some if not all users will be receiving less than their sole optimum catch as their share of the catch. Each share will be determined by the relative weights given the k objectives.

COROLLARY 3.2

The assumptions of Theorem 3.2 *imply that an equitable policy also minimizes the total group regret and minimizes the maximum regret.*

Proof. Same as that for Corollary 3.1. ■

IV. DISCUSSION AND EXTENSIONS

Theorem 2.1 presents a methodology for, and Theorems 3.1 and 3.2 describe efficient policies for, a class of multiple objective problems related to consumption or harvesting models. Traditionally in economics, conflicting objectives or allocations have been treated by pricing mechanisms. Unfortunately, many of the objectives in managing natural resources are difficult to price, but a utility function or preference ordering is often easy to define.

It is also clear that for any efficient policy chosen, the weights λ^i act as relative prices of the different objectives of the user groups. Each efficient policy has an imputed relative price, which can then be determined. The knowledge of the imputed relative prices may cause the decisionmaker to reconsider the decision; even if not, the methodology of this paper avoids some of the artificial pricing schemes that have been developed to evaluate natural resources.

Theorem 3.2 as stated is more appropriate for allocation among groups of users than for allocation among individual harvesters or firms. In the latter case, capacity constraints would more readily enter the model. As long as the concavity assumptions of Sec. III remain valid, and only simple upper and lower bounds are included for each firm, then the results of Theorem 3.2 can be extended readily to the level of the firm.

For many harvesting models, effort rather than catch is the decision variable. For a reasonable set of assumptions, this difference is more apparent than real. Let E be the total effort. Assume total catch is a function of total effort, i.e., $z=h(x, E)$, where $h(\cdot, E)$ is nondecreasing and continuous and $h(x, \cdot)$ is nondecreasing and continuous, and when appropriate, $z^i=(E^i/E)h(x, E)$, that is, the ith user's catch is proportional to its proportion of the total fishing effort. Finally, assume the one-period return satisfies (i)–(iii) in terms of the catch and population size, perhaps for the more general form $g^i(x, y)$.

For Theorem 3.1, note that the catch $z=h(x, E)$, so that y_t is bounded by $h(x, E_{\min}) \leqslant y_t \leqslant h(x, E_{\max})$, where $y_t=x_t-h(x_t, E_t)$. The analysis then proceeds as in Theorem 3.1. For the problem of allocating the catch as in Theorem 3.2, note that $z=h(x, E)$, and $z^i=(E^i/E)h(x, E)$. A similar argument to that in Theorem 3.2 shows that the total catch is bounded by $h(x, E_{\min}) \leqslant z \leqslant h(x, E_{\max})$, and that given the total catch (i.e., given the total effort) and the ith group's effort, then $z^i=(E^i/E)z$, or conversely, given the total catch and the catch of the ith user group, $E^i=E(z^i/z)$.

The above establishes a one to one correspondence between the results about the catch and the results for some models based on effort. The assumptions relating catch and effort are standard; see for example Anderson [1] or Ricker [12].

A major weakness of Theorem 3.2 is that while the user groups fish on the same stock, the total catch does not influence the return to the ith group, that is, the groups either do not supply the same market, or else have no effect on the price each receives. A more realistic assumption would be that each user group's return is of the form

$$p^i\left(\sum_{j=1}^{k} z^j\right)\cdot z^i,$$

so that the total catch affects the price received. I conjecture that as long as concavity is maintained, most of the stated results should be valid for this model.

APPENDIX

Proof of Theorem 2.1. The inductive proof follows the standard proof of stochastic dynamic programming, and hence will only be outlined. At $n=1$, the proof is straightforward.

As an inductive hypothesis, assume the theorem is true in periods $1,2,\ldots,\ n-1$. Then at period n, π_n^* is efficient only if there is no π_n such that

$$\omega_{T-n+1}+v_{\pi_n}(H_{T-n+1}) \geqslant \omega_{T-n+1}+v_{\pi_n^*}(H_{T-n+1}),$$

where ω_{T-n+1} is the value of H_{T-n+1}. Since (ii) is valid at all periods, this implies

$$v_{\pi_n}(x_n) \geqslant v_{\pi_n^*}(x_n),$$

which proves (i).

If at period n, π_{n-1} is not efficient but $\pi_n(y,\pi_{n-1})$ is efficient, define a new policy $\pi_n^*=(y,\pi_{n-1}^*)$ where π_{n-1}^* is an efficient policy. Then π_n^* clearly dominates π_n, contradicting the assumed efficiency of π_n. Combining results yields the desired system of recursive equations. ■

Proof of Theorem 3.1. The proof is by induction on k, first showing that the theorem is valid at $k=2$, and then showing that if the theorem is valid at $k=l$, then it is valid at $k=l+1$.

$k=2$. From [4], it follows that $A_n(x,\lambda)$ is continuous in λ for fixed x, for each n and all $x\in X$. Since at $\lambda=0$ we have $A_n(x,0)=A_n^1(x)$, and at $\lambda=1$ we have $A_n(x,1)=A_n^2(x)$, continuity implies the entire interval $[A_n^1(x), A_n^2(x)]$ must be optimal for some value of λ, $0<\lambda<1$. It remains to show that no value outside the interval is optimal for some value of λ.

Assume that at $\bar{\lambda}$, $0<\bar{\lambda}<1$, we have $A_n(x,\bar{\lambda})<A_n^1(x)$. Since $A_n(x,\cdot)$ is continuous on $[0,1]$, from the definition of continuity, for any $\lambda\in[0,1]$,

$$|A_n(x,1)-A_n(x,\lambda)|<A_n^2(x)-A_n^1(x),$$

since the distance between $A_n(x,1)$ and $A_n(x,0)$ is $A_n^2(x)-A_n^1(x)$. If $A_n(x,\bar{\lambda})$ were less than $A_n^1(x)$, the inequality would be violated, contradicting the continuity of $A_n(x,\cdot)$.

A similar argument holds for $A_n(x,\lambda)>A_n^2(x)$, by using the continuity of $A_n(x,\lambda)$ at $\lambda=0$.

$k=l+1$. Suppose the theorem is valid for $k=l$. The induction is to show that for $k=l+1$, the problem can be transformed into an equivalent two-objective problem.

Let $\gamma=(\gamma^1, \gamma^2, \ldots, \gamma^l)$ be such that $0 \leqslant \gamma^i \leqslant 1$, $i=1, \ldots, l$, $\Sigma_{i=1}^l \gamma^i = 1$. Consider the weighted objective

$$\lambda\left(\sum_{i=1}^{l} \gamma^i g^i(z)\right)+(1-\lambda) g^{l+1}(z). \tag{3.1}$$

By appropriate choice of λ and γ, this weighted objective can be made to take on all values as if an $(l+1)$-vector λ were chosen with $0 \leqslant \lambda^i \leqslant 1$, $\Sigma_{i=1}^{l+1} \gamma^i = 1$. By the induction hypothesis, the optimum for the l objectives must lie between the largest and smallest. From above, the $k=2$ case has the same property, and the $(l+1)$-objective problem has been transformed into an equivalent two-objective problem in (3.1).

Theorem 2.1 proves part (ii). ∎

Proof of Theorem 3.2. By Lemma 3.1, all efficient policies must solve, for some value of λ,

$$f_0(\cdot) \equiv 0,$$

$$f_n(x)=\max_{\Sigma_i z^i \leqslant x}\left\{\sum_{i=1}^{k} \lambda^i g^i(z^i)+\alpha E f_{n-1}\left(s\left[x-\sum_{i=1}^{k} z^i, d\right]\right)\right\}.$$

The proof uses a transformation introduced by Karush [7] and later exploited by Veinott [13]. Let $w^1=z^1$, $w^2=z^2+z^1, \ldots, w^k=\Sigma_{i=1}^k z^k$. The recursion becomes

$$f_n(x)=\max_{\substack{0 \leqslant w^1 \leqslant w^2 \\ w^2 \leqslant w^3 \leqslant w^4 \\ \vdots \\ w^{k-1} \leqslant w^k \leqslant x}}\left\{\sum_{i=1}^{k} \lambda^i g^i(w^i-w^{i-1})+\alpha E f_{n-1}(s[x-w^k, d])\right\}$$

Assume $w^2, w^3, \ldots, w^k$ are fixed. Then the optimum for w^1 is to

$$\underset{0 \leqslant w^1 \leqslant w^2}{\operatorname{maximize}} \quad \lambda^1 g^1(w^1)+\lambda^2 g^2(w^2-w^1)=h_1(w^2).$$

Let $A_1(w^2)$ be the solution of $h_1(w^2)$ for each w^2. Then by standard dynamic programming arguments as in [11], $h_1(w^2)$ is concave, continuous, and nondecreasing in w^2. This argument can be repeated till the recursion can be written as

$$\begin{aligned} f_n(x)= \max_{0 \leqslant w^k \leqslant x} & \{h_{k-1}(w^k)+\lambda^k g^k(w^k-A_{k-1}(w^k)) \\ & +\alpha E f_{n-1}(s[x-w^k, d])\} \end{aligned}$$

where $h_{k-1}(w^k)$ is concave, continuous, and nondecreasing, and $g^k(w^k - A_{k-1}(w^k))$ is concave in w^k. This proves (ii), as the problem has been transformed into one that depends only on w^k, and an allocation given w^k that solves (3.2). The proof of (i) again is by induction on k.

$k=2$. From [4], an optimal solution is continuous in λ on $[0,1]$. Since at $\lambda=0$ an optimal total amount to be harvested is $z_n^{2*}(x)$, and at $\lambda=1$ an optimal total amount to be harvested is $z_n^{1*}(x)$, then continuity implies every total harvest in the interval $[z_n^{1*}(x), z_n^{2*}(x)]$ must also be part of an efficient policy. Therefore, it remains to be proved that no total harvest outside of the interval is part of an efficient policy. However, the same continuity arguments as those in Theorem 3.1 imply this result.

$k=l$. The inductive argument is the same as that in Theorem 3.1. Theorem 2.1 proves part (iii). ■

REFERENCES

1 L. G. Anderson, Optimum economic yield of an internationally utilized common property resource, *U.S. National Marine Fisheries Service*, *Fishery Bulletin* 73:51–66 (1975).

2 T. A. Brown and R. G. Strauch, Dynamic programming in multiplicative lattices, *J. Math. Anal. Appl.* 12:364–370 (1964).

3 E. V. Denardo, Contraction mappings in the theory underlying dynamic programming, *SIAM Rev.* 9:165–177 (1967).

4 A. M. Geoffrion, Solving bicriterion mathematical programs, *Operations Res.* 15:39–54 (1967).

5 ———, Proper efficiency and the theory of vector maximization, *J. Math. Anal. Appl.* 22:618–630 (1968).

6 M. I. Henig, Multicriteria dynamic programming, Ph.D. Dissertation, Yale Univ., New Haven, 1978, 75 pp.

7 W. Karush, A theorem in convex programming, *Nav. Res. Logist. Quart.* 6:245–260 (1959).

8 O. L. Mangasarian, *Nonlinear Programming*, McGraw-Hill, New York, 1969, 220 pp.

9 R. Mendelssohn, Optimal harvesting strategies for stochastic populations, Ph.D. Dissertation, Yale Univ., New Haven, 1976, 98 pp.

10 ———, Harvesting with smoothing costs, Southwest Fisheries Center Honolulu Laboratory, National Marine Fisheries Service, NOAA, Administrative Report 9H, 1977, 23 pp.

11 R. Mendelssohn, and M. J. Sobel, Capital accumulation and the optimization of renewable resource models, *J. Economic Theory*, to appear.

12 W. E. Ricker, Computation and interpretation of biological statistics of fish populations, *Fisheries Research Board of Canada*, *Bulletin*, 191:382 (1975).

13 A. F. Veinott, Jr., The status of mathematical inventory theory, *Management Sci.* 12:745–777 (1966).

14 P. L. Yu, and G. Leitmann, Compromise solutions, domination structures, and Salukuadze's solution, in *Multicriteria Decision Making and Differential Games*, (G. Leitmann, Ed.), Plenum, New York, 1976, pp. 85–102.

15 W. I. Zangwill, An algorithm for the Chebyshev problem—with an application to concave programming, *Management Sci.* 14:58–78 (1967).